In mathematics, **pentation** (or **hyper-5**) is the fifth hyperoperation. Pentation is defined to be repeated tetration, similarly to how tetration is repeated exponentiation, exponentiation is repeated multiplication, and multiplication is repeated addition. The concept of "pentation" was named by English mathematician Reuben Goodstein in 1947, when he came up with the naming scheme for hyperoperations.

The number *a* pentated to the number *b* is defined as *a* tetrated to itself *b* - *1* times. This may variously be denoted as $a[5]b$, $a \uparrow\uparrow\uparrow b$, $a \uparrow^3 b$, $a \to b \to 3$, or ${}_b\square a$, depending on one's choice of notation.

For example, 2 pentated to 2 is 2 tetrated to 2, or 2 raised to the power of 2, which is $2^2=4$. As another example, 2 pentated to 3 is 2 tetrated to the result of 2 tetrated to 2. Since 2 tetrated to 2 is 4, 2 pentated to 3 is 2 tetrated to 4, which is $2^{2^{2^2}}=65536$.

Based on this definition, pentation is only defined when *a* and *b* are both positive integers, though progress has been made to allow for any value of *a*.

Definition

Pentation is the next hyperoperation (infinite sequence of arithmetic operations, based on the previous one each time) after tetration and before hexation. It is defined as iterated (repeated) tetration (assuming right-associativity). This is similar to tetration, as tetration is iterated right-associative exponentiation.[1] It is a binary operation defined with two numbers *a* and *b*, where *a* is tetrated to itself $b - 1$ times.

The type of hyperoperation is typically denoted by a number in brackets, []. For instance, using hyperoperation notation for pentation and tetration, $2[5]3$ means tetrating 2 to itself 2 times, or $2[4](2[4]2)$. This can then be reduced to $2[4](2^2)=2[4]4=2^{2^{2^2}}=2^{2^4}=2^{16}=65{,}536$.

Etymology

The word "pentation" was coined by Reuben Goodstein in 1947 from the roots penta- (five) and iteration. It is part of his general naming scheme for hyperoperations.[2]

Notation

There is little consensus on the notation for pentation; as such, there are many different ways to write the operation. However, some are more used than others, and some have clear advantages or disadvantages compared to others.

- Pentation can be written as a hyperoperation as $a[5]b$. In this format, $a[3]b$ may be interpreted as the result of repeatedly applying the function $x \mapsto a[2]x$, for b repetitions, starting from the number 1. Analogously, $a[4]b$, tetration, represents the value obtained by repeatedly applying the function $x \mapsto a[3]x$, for b repetitions, starting from the number 1, and the pentation $a[5]b$ represents the value obtained by repeatedly applying the function $x \mapsto a[4]x$, for b repetitions, starting from the number 1.[3][4] This will be the notation used in the rest of the article.
- In Knuth's up-arrow notation, $a[5]b$ is represented as $a \uparrow\uparrow\uparrow b$ or $a \uparrow^3 b$. In this notation, $a \uparrow b$ represents the exponentiation function a^b and $a \uparrow\uparrow b$ represents tetration. The operation can be easily adapted for hexation by adding another arrow.
- In Conway chained arrow notation, $a[5]b = a \to b \to 3$.[5]
- Another proposed notation is $\text{❑}_b a$, though this is not extensible to higher hyperoperations.[6]

Examples

The values of the pentation function may also be obtained from the values in the fourth row of the table of values of a variant of the Ackermann function: if $A(n, m)$ is defined by the Ackermann recurrence $A(m-1, A(m, n-1))$ with the initial conditions $A(1, n) = an$ and $A(m, 1) = a$, then $a[5]b = A(4, b)$.[7]

Although tetration, its base operation, was extended to non-integer heights (more generally any height, real or complex) in 2017,[8] building on Hellmuth Kneser's work,[9] pentation $a[5]b$ is currently only defined for integer values of b where $a > 0$ and $b \geq -2$, such as $e[5]2 = e[4]e \approx 2075.96834\ldots$. As with all hyperoperations of order 3 (exponentiation) and higher, pentation has the following trivial cases (identities) which holds for all values of a and b within its domain:

- $1[5]b = 1$

- $a[5]1=a$

Additionally, we can also introduce the following defining relations:

- $a[5]2=a[4]a$
- $a[5]0=1$
- $a[5](-1)=0$
- $a[5](-2)=-1$
- $a[5](b+1)=a[4](a[5]b)$

Other than the trivial cases shown above, pentation generates extremely large numbers very quickly. As a result, there are only a few non-trivial cases that produce numbers that can be written in conventional notation, which are all listed below.

Some of these numbers are written in power tower notation due to their extreme size. Note that $\exp_{10}(n)=10^n$.

- $2[5]2=2[4]2=2^2=4$
- $2[5]3=2[4](2[5]2)=2[4](2[4]2)=2[4]4=2^{2^{2^2}}=2^{2^4}=2^{16}=65{,}536$
- $2[5]4=2[4](2[5]3)=2[4](2[4](2[4]2))=2[4](2[4]4)=2[4]65{,}536=2^{2^{2^{\cdot^{\cdot^{\cdot}}}}}$ (a power tow
- $2[5]5=2[4](2[5]4)=2[4](2[4](2[4](2[4]2)))=2[4](2[4](2[4]4))=2[4](2[4]65{,}536)=2$
- $3[5]2=3[4]3=3^{3^3}=3^{27}=7{,}625{,}597{,}484{,}987$
- $3[5]3=3[4](3[5]2)=3[4](3[4]3)=3[4]7{,}625{,}597{,}484{,}987=3^{3^{3^{\cdot^{\cdot^{\cdot}}}}}$ (a power tower of h
- $3[5]4=3[4](3[5]3)=3[4](3[4](3[4]3))=3[4](3[4]7{,}625{,}597{,}484{,}987)=3^{3^{3^{\cdot^{\cdot^{\cdot}}}}}$ (a power
- $4[5]2=4[4]4=4^{4^{4^4}}=4^{4^{256}}\approx\exp_{10}^3(2.19)$ (a number with over 10^{153} digits)
- $5[5]2=5[4]5=5^{5^{5^{5^5}}}=5^{5^{5^{3125}}}\approx\exp_{10}^4(3.33928)$ (a number with more than $10^{10^{2184}}$ digits)

See also

- Ackermann function
- Large numbers
- Graham's number
- History of large numbers